Main Results and Properties of Fractional Calculus

Rajesh Kumar Shukla[1*] & Puneet Sapra[2]
Department of Applied Sciences and Humanities
Invertis University, Bareilly-243 123, U.P., India

Abstract
In this paper our aim is to discuss the main results and properties of the fractional calculus like fractional integrals and derivatives of the non-integer order or arbitrary order. And we will also discuss about those functions which are used in fractional calculus namely Gamma function, Mittag-Leffer function, Agarwal function, Erdelyi's function etc.

Subject Classification: Mathematics 2000: 26A33,33E12,33E20,44A20,45E10,45J05.

Keywords: *fractional derivative, fractional integral, differintegrals, Caputo fractional derivative*

1.Introduction

The first question that comes to mind is how the fractional calculus came into existence and the answer probably lies within the conversation of Leibniz and G.A. L'Hopital in the early seventeenth century. The things started when L'Hopital wrote a letter dated September 30, 1965 to Leibnitz asking him about a particular notation that he had used in his paper for the n^{th} derivative of a function i.e.

$$\frac{d^n f(x)}{dx^n} \quad or\ D^n\,\{f(x)\}, \qquad D \equiv \frac{d}{dx}$$

L'Hopital wanted to know the result, when $n = {}^1\!/_2$ means he was interested to find out the derivative of order $\frac{1}{2}$. G.W.Leibniz wrote, prophetically, "Thus it follows that $d^{\frac{1}{2}}\,x$ will be equal to $x\sqrt[2]{dx:x}$,an apparent paradox, from which one day useful consequences will be drawn (Leibniz,1962)." In these words, by Leibniz probably the fractional calculus was born. Leibniz wrote another letter to J. Bernoulli and in this letter he mentioned the term derivative of the general order. In fact Leibniz discussed the infinite product of Wallis for π and used the notation $d^{\frac{1}{2}}\,y$ to denote the derivative of order $\frac{1}{2}$ [2]. In 1730, L.Euler mentioned in his work that when n is a positive integer, the ratio of $d^n p, p$ being a function of x and dx^n can always be expressed algebraically. Even he suggested to use the interpolation series if n is fraction. But if n is a positive integer then d^n can be found by continued differentiation. In 1772, J.L. Lagrange had developed the law of exponent (indices) for differential operators of integer orders:

$$\frac{d^m}{dx^m}\frac{d^n}{dx^n}\,y \quad = \frac{d^{m+n}}{dx^{m+n}}\,y.$$

It was very important to know when the theory of fractional calculus started that whether this law held true if $m\ and\ n$ were fractions. In 1882, P.S. Laplace wrote expressions for a derivative of non-integer orders by using the integrals. S.F.Lacroix developed a formula for fractional differentiation for the n^{th} derivative of x^m by induction. He formally obtained the derivative of order $\frac{1}{2}$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\, x = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

Where $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. He obtained the above expression in the year 1819.

In 1822, J.B.J. Fourier obtained the integral formula for the derivative of order u which is as follows:

$\frac{d^u}{dx^u} f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\beta) d\beta \int_{-\infty}^{+\infty} p^u \cos\left(px - p\beta + \frac{u\pi}{2}\right) dp$,

where u is any quantity i.e. positive or negative. The above result can be derived in a more suitable way as suggested by Liouville in the year 1835. The tautochrone (isochrone) problem was studied by N.H. Abel in 1823 (This paper first appear in Mag. Naturvidenkaberne) (Abel,1823). He used derivative of arbitrary order to solve the problem. The following is the integral

$$\int_0^y (y-t)^{\frac{1}{2}} f(y) dt$$

Abel worked with this integral. This result is considered the first application of the fractional calculus.

1.1 A Generalization of Integer Order Calculus

When we talk about any quantity which in the form of x^n then easily we visualize that x is multiply $by\ n$ times will give the required result. In this case we assume that n is an integer. But the question is if n is not an integer then how we will visualize it. For example visualize 3^e , the answer is, it is difficult to visualize but its result exists i.e. $3^e = 19.812990$. In the same way the fractional derivative

$$\frac{d^e}{dx^e} f(x)$$

is not easy to visualize like x^n and moreover presently does not exist. As we are aware that the real numbers exist between the integers but the question is does the fractional differintegrals do exist between conventional integer order derivatives and n folds integrations. Now we see the generalization from integer to real number on number line as follows:

$x^n = x.\ x.\ x \dots.\ x,$ $\qquad x\ is\ multiplied\ by\ n\ times\ where\ n\ is\ an\ integer$

$n! = 1.2.3 \dots.n\,(n-1),$ $\qquad where\ n\ is\ an\ integer$

$n! = n.(n-1)! = \Gamma(n+1) = n.\Gamma n,$ $\qquad where\ n\ is\ real$

The above result shows that the generalization from integer to non-integer, so in general we are making number line.

The fractional calculus can be understood precisely by knowing some of the simple mathematical functions which are used in fractional calculus or in other words these functions are the basic functions for the theory of fractional calculus. These have been discussed in the following subsections.

2. Functions Used in Fractional Calculus

In this section our aim is to discuss a number of functions which are useful in providing the solutions to the problems of fractional calculus. R.K. Shukla and P. Sapra (2019) recently published a paper on "Fractional Calculus and its Applications for scientific professionals: A literature review. In this paper the authors had discussed in details about the main functions, properties and applications of the fractional calculus.

2.1 The Gamma Function

This function is the base function or main function. The Gamma function generalizes the factorial function used in multiple differentiation and repeated integrations in integer order calculus. Euler's Gamma function is defined as follows:

$$\Gamma z = \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

Here considering z as real number. So the above result implies that the Gamma function is defined for positive real values of z.

Some standard results of the Gamma function are as follows:

$$\Gamma 1 = 1, \qquad \Gamma \frac{1}{2} = \sqrt{\pi}$$
$$\Gamma(z+1) = z\Gamma z$$
$$\Gamma 2 = 1\Gamma 1 = 1 = 1!$$
$$\Gamma 3 = 2\Gamma 2 = 2 = 2!$$
$$\Gamma 4 = 3\Gamma 3 = 6 = 3!$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Gamma(n+1) = n\Gamma n = n(n-1)! = n!$$

The above result is valid for the positive values of z. Apart from the above results, the Gamma function has another important results that this function has simple poles at $z = 0, -1, -2, -3, \ldots$ The proof is explained on page no. 20 &21 of Shantanu Das (Das,2008).

2.2 Mittag-Leffer Function of One-Parameter

In1903, Mittag-Leffer introduced the one-parameter function and it is defined as follows:

$$E_\alpha(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + 1)}$$

Or

$$E_\alpha(z) = 1 + \frac{z}{\Gamma(\alpha+1)} + \frac{z^2}{\Gamma(2\alpha+1)} + \cdots$$

2.3 Mittag-Leffer Functions of Two-Parameter

R.P. Agarwal and Erdelyi introduced the two-parameter Mittag-Leffer functions during 1953-1954. This function is defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + \beta)} \ , \quad \alpha > 0, \beta > 0$$

One –Parameter Mittag-Leffer function can be obtain by taking $\beta = 1$, i.e.
:

$$E_{\alpha,1}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + 1)} = E_\alpha(z)$$

Exponential function (e^z) plays an important role in conventional calculus i.e. integer order calculus equations. Similarly, the Mittag-Leffer functions plays important role in the fractional order calculus.

2.4 The Agarwal Function

This function is defined as follows:

$$E_{\alpha,\beta}\,(t) = \sum_{m=0}^{\infty} \frac{t^{\left(\frac{\beta-1}{\alpha}+m\right)}}{\Gamma(\alpha m+\beta)}$$

In the fractional order system theory, the Agarwal function is very much useful because of its Laplace transform, it was obtained by Agarwal i.e.

$$L\{E_{\alpha,\beta}(t^{\alpha}) \;=\; \frac{s^{\alpha-\beta}}{s^{\alpha}-1}$$

In addition to this, R.P. Agarwal has generalized the Mittag-Leffer function in 1953.

2.5 The Rovotnov-Hartley Function

This function is defined as follows:

$$F_q(-a,t) = t^{q-1}\sum_{n=0}^{\infty} \frac{(-a)^n t^{nq}}{\Gamma q(n+1)}\,, \quad q>0$$

This function was introduced by Robotnov and Hartley in 1988. The direct solution of the fundamental linear fractional order differential equations is affected by this function. One of the important parts of this function is the power and simplicity of its Laplace transform, particularly

$$\mathcal{L}\{F_q(a,t)\} = \frac{1}{s^q-1}\,, \qquad q>0$$

2.6 The Miller-Ross' Function

In the year 1993, Miller and Ross introduced a function as the basis of the solution of the fractional order initial value problem. The function is defined as the vth integral of the exponential function, which is in the following form (Miller and Ross,1993).

$$E_t(v,a) = \frac{d^{-v}}{dt^{-v}} = t^{v}\sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(v+k+1)} = t^{v}\, e^{at}\,\gamma^{*}(v,at),$$

where $\gamma^{*}(v,at)$ is the incomplete gamma function. The incomplete Gamma function is a closely related function defined as

$$\gamma^{*}(v,t) = \frac{t^{-v}}{\Gamma v}\int_0^t e^{-x}\,x^{v-1}\,dx, \;\; v>0$$

The Laplace transform of the function can be written as

$$\mathcal{L}\{E_t(v,a)\} = \frac{s^{-v}}{s-a}\,, \quad R_e(v)>1.$$

This function $E_t(v,a)$ is closely related to incomplete Gamma and Mittag-Leffer functions. So, it is defined as

$$E_t(v,a) = t^{v}\, e^{at}\,\gamma^{*}(v,t)$$

2.7 The Generalized R and G Function

In the analysis of fractional order differential equations, this function has great importance. This function is defined as follows:

$R_{q,v}[a,t,c] = \sum_{n=0}^{\infty} \frac{(a)^n (t-c)^{(n+1)q-1-v}}{\Gamma\{q(n+1)-v\}}$,

where t is independent variable and c is the lower limit of fractional differintegration. Our interest in this function will be for the solution of fractional differential equations for the range of $t > c$. The more compact notation is as follows:

$R_{q,v}[a,t-c] = \sum_{n=0}^{\infty} \frac{(a)^n (t-c)^{(n+1)q-1-v}}{\Gamma\{q(n+1)-v\}}$

It is useful, particularly when $c = 0$.

2.8 The Wright Function

This function is defined as follows:

$W(x;\alpha,\beta) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r+\beta)\, r!} \quad , \quad \alpha > 0, \ \beta > 0$

where $x \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$.

2.9 The Mainardi Function

This function is defined as follows:

$W(-x;-\alpha,1-\alpha) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(\alpha r+\beta)} \frac{x^r}{r!} \quad , \quad \alpha > 0$

Mainardi function is also denoted by $M(x;\alpha)$

3. The Fractional Integral

The following identities follow from the definition of Mittag-Leffer function of two parameters which is given in the subsection 2.3

$$E_{1,1}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(r+1)} = \sum_{r=0}^{\infty} \frac{z^r}{r!} = e^z$$

$$E_{1,2}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(r+2)} = \sum_{r=0}^{\infty} \frac{z^r}{(r+1)!} = \frac{1}{z}\sum_{r=0}^{\infty} \frac{z^{r+1}}{(r+1)!} = \frac{e^z - 1}{z}$$

In this way we can write, the most general form of the above results as:

$$E_{1,n}(z) = \frac{1}{z^{n-1}} \left(e^z - \sum_{r=0}^{\infty} \frac{z^r}{r!} \right)$$

Number of important identities can be derived but it is up to reader which identity they want.

The Laplace transform technique is very much required for the solution of fractional differintegrals and many authors used this technique. The Laplace transform of a function $f(t)$ can be written as:

$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s)$ (1)

There are number of important properties of Laplace transform but here we have mentioned only two properties which are useful for our case. The first one is the Laplace transform of the derivative of order n i.e.

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)$$

The result (24) shows that $1/2^{th}$ order derivative of the constant function is not zero.

Example 5.3.2

To find the fractional derivative of e^{ax} $(or\ D^{\alpha}\ (e^{ax}))$ of order α, where $0 < \alpha < 1$.
We are applying the formula (22) instead of (23) because (22) is more suitable in this case. So we get the following:

$$D^{\alpha}e^{ax} = D^{1}\{D^{-\alpha}e^{ax}\}$$

Using (18), then we get

$$D^{\alpha}e^{ax} = D^{1}\{x^{\alpha}E_{1,\alpha+1}(ax)\}$$

Now using the definition of Mittag-Leffer functions (2.1) & (2.2), we get the following result
$D^{\alpha}e^{ax} = x^{-\alpha}E_{1,-\alpha+1}(ax)$.

6. Recent Work

In year 2019, G. Sales Teodoro, J.A. Tenreiro Machodo and Edmundo Capelas de Oliveira have published their work on "A review of definitions of fractional derivatives and other operators" (Sales Teodoro et al.,2019). In this paper authors have reviewed the fractional derivatives definitions and operators used in it and it has been found that the two popular approaches i.e. Riemann-Liouville and Caputo are the most fitted definitions for the fractional derivatives. A number of books and research papers have been published recently and in this year, Edmundo Capelas de Oliveira has published a book "Solved Exercises in Fractional Calculus, Springer Nature Switzerland AG, 2019.This book has covered latest development in the field of fractional calculus and its applications and also given the historical survey from 1695 to 2019 (Capelas de Oliveira, 2019).

7. Conclusion

In this paper we discussed the main functions, properties, standard results for the fractional integrals and derivatives. We have seen that how some mathematical functions like Gamma function, incomplete Gamma function and Beta function have played important role in finding the fractional integrals and derivatives. We also discussed number of functions like Mittag-Leffer functions of one and two parameters, Agarwal function, etc. As we are aware that a number of definitions are available for the fractional integrals and derivatives but the two main or most popular definitions are Riemann-Liouville and Caputo. In Example 5.1.1, we have established an important result by equation (17) and this equation helps us to find out the fractional integral of a constant function and it is found to be not zero in general. In (24), we have seen that the $1/2th$ order derivative of the constant function is not zero in general where as in differential calculus of positive integer, the derivative of the constant function is always zero. Fractional calculus is such a field where many models are still need to be introduced and applied in real world applications in many areas of science and engineering and in this direction a research paper (Sun et al., 2018) entitled" A new collection of real world applications of fractional calculus in science and engineering" was published.

References:

*[1] N.H. Abel, "Solution de quelques problemes a l'aide d'integrals definites Oeuvres Completes". Grondahl, Christiania, Norway. vol. 81, no. 1 (**1823**), pp 16-18.*

*[2] C. de Oliveira, E. J. Vaz Jr, and S. Jarosz, "Fractional Calculus via Laplace transform and its application in relaxation processes", Commun. Nonlinear Sci. Numer. Simulat, vol. 69 **(2019)**, pp.58-72.*

*[3] C. de Oliveira, E. J. Vaz Jr, "Solved Exercises in Fractional Calculus". Springer Nature Switzerland, **(2019)**.*

*[4] S. Das, "Functional Fractional Calculus for System of Identification and Controls", Springer-Verlag Berlin Heidelberg, **(2008).***

*[5] R. Gorenflo and F Mainardi, "Fractional Calculus: Integral and Differential Equations of Fractional Order", Springer Verlag Wien and New York, (**1997**), pp. 223-276.*

*[6] J. M. Kimeu, "Fractional Calculus: Definition and Applications", Western Kentucky University, Top SCHOLAR, **(2009).***

*[7] V. Krupkova and Z Samarda, "On some properties on fractional calculus, ICSC-Fourth International Conference on Soft Computing Applied in Computer and Economic environment", EPI Kunoice, Czech Republic, **(2006)**.*

[8] G.W. Leibniz, "Letter from Hanover, Germany" September 30, 1695 to G.A. L'Hopital, Leibnizen Mathematische Schriften. Olms Verlag., Hildesheim, Germany, 2, 301-302.

[9] Leibniz, G.W. (1962). Letter from Hanover, Germany, May 28, 1697 to J. Wallis, Leibnizen Mathematische Schriften. Olms Verlag., Hildesheim, Germany, 4, 25.

*[10] A. M. Mathai, and H.J Haubold, "Special functions for applied scientists", Springer. **(2008)**.*

*[11] K. Miller and B Ross, "An introduction to the fractional calculus and fractional differential equations" John Wiley & Sons, Inc. **(1993).***

*[12] I. Podlubny, "Fractional Differential Equations. Mathematics in Science and Engineering" Academic Press **(1999).***

*[13] G. Sales Teodoro, J.A Tenreiro Machado and C. de Oliveira, E. J. Vaz Jr, "A review of definitions of fractional derivatives and other operators", J. Comput. Phys., vol. 388, **(2019)** pp. 195-208.*

*[14] R.K Shukla and P. Sapra, "Fractional Calculus and Its Applications for Scientific Professionals: A Literature Review", International Journal of Modern Mathematical Sciences USA, vol. 17 np. 2, **(2019)**, pp. 111-137.*

*[15] H.G Sun, Y. Zhang, D. Baleanu, W. Y. Q Chen, "A new collection of real world applications of fractional calculus in science and engineering", Communications in Nonlinear Science and Numerical Simulation, vol. 64, **(2018)**, pp. 213-231.*